# A Hybrid Intuitionistic Logic: Semantics and Decidability

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ABSTRACT. An intuitionistic, hybrid modal logic suitable for reasoning about distribution of resources was introduced in [16, 17]. The modalities of the logic allow to validate properties in a *particular place*, in *some* place and in *all* places. We give a sound and complete Kripke semantics for the logic extended with disjunctive connectives. The extended logic can be seen as an instance of *Hybrid IS5*. We also give a sound and complete birelational semantics, and show that it satisfies the finite model property: if a judgement is not valid in the logic, then there is a finite birelational counter-model. Hence we prove that the logic is decidable.

## Contents



## 1 Introduction

In current computing paradigm distributed resources spread over and shared amongst different nodes of a computer system are very common. For example, printers may be shared in local area networks, or distributed data may store documents in parts at different locations. The traditional reasoning methodologies are not easily scalable to these systems as they may lack implicitly trust-able objects such as a central control.

This has resulted in the innovation of several reasoning techniques. A popular approach in the literature has been the use of algebraic systems such as process algebra [10, 20, 15]. These algebras have rich theories in terms of semantics [20], logics [9, 8, 14, 22], and types [15]. Another approach is logic-oriented [16, 17, 37, 21, 38, 30]: intuitionistic modal logics are used as foundations of type systems by exploiting the *propositions-as-types*, *proofs-as-programs* paradigm [12]. An instance of this was introduced in [16, 17]. The logic introduced there is the focus of our study. It uses the conjunctive connectives □ and □, and implication $\rightarrow$.

The formulae in this logic also include names, called *places*. Assertions in the logic are associated with places, and are validated in places. In addition to considering *whether* a formula is true, we are also interested in *where* a formula is true. In order to achieve this, the logic has three modalities. The modalities allow us to infer whether a property is validated in a specific place of the system ($@p$), or in an unspecified place of the system ($\blacklozenge$), or in any part of the system (□

The deduction system is essentially a conservative extension of propositional intuitionistic logic; and it is in this sense that we will use the adjective "intuitionistic" for the extended logic throughout the paper.

As noted in [16, 17], the logic can also be used to reason about distribution of resources in addition to serving as the foundation of a type system. The papers [16, 17], however, lack a model to match the usage of the logic as a tool to reason about distributed resources. In this paper, we bridge the gap by presenting a Kripke-style semantics [19] for the logic extended with disjunctive connectives. In Kripke-style semantics, formulae are considered valid if they remain valid when the atoms mentioned in the formulae change their value from false to true. This is achieved by using a partially ordered set of *possible states*. Informally, more atoms are true in larger states.

We extend the Kripke semantics of the intuitionistic logic [19], enriching each possible state with a set of places. The set of places in Kripke states are not fixed, and different possible Kripke states may have *different* set of places. However, the set of places vary in a conservative way: larger Kripke states contain larger set of places. In each possible state, different places satisfy different formulae. In the model, we interpret atomic formulae as resources of a distributed system, and placement of atoms in a possible state corresponds to the distribution of resources.

The enrichment of the model with places reveals the true meaning of the modalities in the logic. The modality $@p$ expresses a property in a named place. The modality $\Box$ corresponds to a weak form of spatial universal quantification and expresses a property common to all places, and the modality $\Diamond$ corresponds to a weak form of spatial existential quantification and expresses a property valid somewhere in the system. For the intuitionistic connectives, the satisfaction of formulae at a place in a possible state follows the standard definition [19].

To give semantics to a logical judgement, we allow models with more places than those mentioned in the judgement. This admits the possibility that a user may be aware of only a certain subset of names in a distributed system. This is crucial in the proof of soundness and completeness as it allows us to create witnesses for the existential ($\Diamond$) and the universal ($\Box$) modalities. The Kripke semantics reveals that the extended logic can be seen as the hybridisation of the well-known intuitionistic modal system *IS5* [11, 23, 26, 29, 34, 35].

Following [11, 26, 34, 35], we also introduce a sound and complete birelational semantics for the logic. The reason for introducing birelational semantics is that it allows us to prove decidability. Birelational semantics typically enjoy the *finite model property*

and $\mathsf{doc}_2$ are stored in a particular place, then the usual intuitionistic rules allow to infer that the place can access the entire document.

The intuitionistic framework is extended in [17] to reason about different places. An assertion in such a logic takes the form "$\phi$ ***at*** $p$", meaning that formula $\phi$ is valid at place $p$. The construct "**at**" is a meta-linguistic symbol and points to the place where the reasoning is located. For example, $\mathsf{doc}_1$ ***at*** $p$ and $\mathsf{doc}_2$ ***at*** $p$ formalise the notion that the parts $\mathsf{doc}_1$ and $\mathsf{doc}_2$ are located at the node $p$. If, in addition, the assertion $((\mathsf{doc}_1 \wedge \mathsf{doc}_2) \rightarrow \mathsf{doc})$ ***at*** $p$ is valid, we can conclude that the document $\mathsf{doc}$ is available at $p$.

The logic is a conservative extension of intuitionistic logic in the sense that if we restrict our attention to formulae without modalities then the 'local' proof system in a single place $p$ mimics the standard intuitionistic one. For instance, the deduction described above is formally

$$\dfrac{\dfrac{;\Delta \vdash^{\{p\}} \mathsf{doc}_1 \textbf{\textit{ at }} p \qquad ;\Delta \vdash^{\{p\}} \mathsf{doc}_2 \textbf{\textit{ at }} p}{;\Delta \vdash^{\{p\}} \mathsf{doc}_1 \wedge \mathsf{doc}_2 \textbf{\textit{ at }} p}\ \wedge I \qquad ;\Delta \vdash^{\{p\}} (\mathsf{doc}_1 \wedge \mathsf{doc}_2) \rightarrow \mathsf{doc} \textbf{\textit{ at }} p}{;\Delta \vdash^{\{p\}} \mathsf{doc} \textbf{\textit{ at }} p}\ \rightarrow E \tag{1}$$

where $\Delta \stackrel{\text{def}}{=} (\mathsf{doc}_1 \wedge \mathsf{doc}_2) \rightarrow \mathsf{doc}$ ***at*** $p, \mathsf{doc}_1$ ***at*** $p, \mathsf{doc}_2$ ***at*** $p$. It is easy to see that this derivation becomes a standard intuitionistic one if rewritten without the 'place' ***at*** $p$.

In the assertion $\phi$ ***at*** $p$, $\phi$

the section (see Ex. 1).

Even if we deal with resources distributed in heterogeneous places, certain properties are valid everywhere. For this purpose, the logic has the modality $\Box$: the formula $\Box\phi$ means that $\phi$ is valid everywhere. In the example above, $p$ can access the document $\mathsf{doc}$, if there is a place that has the part $\mathsf{doc}_2$ and can send it everywhere. This can be expressed by the formula $\blacklozenge(\mathsf{doc}_2 \wedge (\mathsf{doc}_2 \rightarrow \Box\mathsf{doc}_2))$ ***at*** $p'$. The rules of the logic would allow us to conclude that $\mathsf{doc}_2$ is available at $p$. Therefore the document $\mathsf{doc}$ is also available at $p$. We will illustrate this inference at the end of the section (see Ex. 2).

We now define formally the logic. As mentioned above, it is essentially the logic introduced in [17] enriched with the disjunctive connectives $\bot$ and $\vee$, thus achieving the full set of intuitionistic connectives. This allows us to express properties such as: the document $\mathsf{doc}_2$ is located either at $p$ itself or at $q$ (in which case $p$ has to fetch it). This can be expressed by the formula $(\mathsf{doc}_2 \vee ((\mathsf{doc}_2@q) \rightarrow \mathsf{doc}_2))$ ***at*** $p$.

For the rest of the paper, we shall assume a fixed countable set of atomic formulae *Atoms*, and we vary the set of places. Given a countable set of places *Pl*, let *Frm*(*Pl*) be the set of formulae built from the following grammar:

$$\phi ::= A \mid \top \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid \phi@p \mid \Box\phi \mid \blacklozenge\phi.$$

Here the syntactic category $p$ stands for elements from *Pl*, and the syntactic category $A$ stands for elements from *Atoms*. The elements in *Frm*(*Pl*) are said to be *pure formulae*, and are denoted by small Greek letters $\phi, \psi, \mu \ldots$ An assertion of the form $\phi$ ***at*** $p$ is called *sentence*. We denote by capital Greek letters $\Gamma, \Gamma_1, \ldots$ (possibly empty) finite sets of pure formulae, and by capital Greek letters $\Delta, \Delta_1, \ldots$ (possibly empty) finite sets of sentences.

Each judgement in this logic is of the form

$$\Gamma; \Delta$$

$$\frac{}{\Gamma;\Delta,\phi\ \boldsymbol{at}\ p \vdash^P \phi\ \boldsymbol{at}\ p}\ L \qquad \frac{}{\Gamma,\phi;\Delta \vdash^P \phi\ \boldsymbol{at}\ p}\ G$$

$$\frac{}{\Gamma;\Delta \vdash^P \top\ \boldsymbol{at}\ p}\ \top I \qquad \frac{\Gamma;\Delta \vdash^P \bot\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash^P \psi\ \boldsymbol{at}\ p}\ \bot E$$

$$\frac{\Gamma;\Delta \vdash^P \phi_1\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash^P \phi_1 \vee \phi_2\ \boldsymbol{at}\ p}\ \vee I_1 \qquad \frac{\Gamma;\Delta \vdash^P \phi_2\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash^P \phi_1 \vee \phi_2\ \boldsymbol{at}\ p}\ \vee I_2$$

$$\frac{\Gamma;\Delta \vdash^P \phi_1 \vee \phi_2\ \boldsymbol{at}\ p \quad \Gamma;\Delta,\phi_1\ \boldsymbol{at}\ p \vdash^P \psi\ \boldsymbol{at}\ p \quad \Gamma;\Delta,\phi_2\ \boldsymbol{at}\ p \vdash^P \psi\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash^P \psi\ \boldsymbol{at}\ p}\ \vee E$$

$$\frac{\Gamma;\Delta \vdash^P \phi_i\ \boldsymbol{at}\ p \quad i = 1,2}{\Gamma;\Delta \vdash^P \phi_1 \wedge \phi_2\ \boldsymbol{at}\ p}\ \wedge I \qquad \frac{\Gamma;\Delta \vdash^P \phi_1 \wedge \phi_2\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash^P \phi_i\ \boldsymbol{at}\ p}\ \wedge E_i\ \ (i = 1,2)$$

$$\frac{\Gamma;\Delta,\phi\ \boldsymbol{at}\ p \vdash^P \psi\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash^P \phi \to \psi\ \boldsymbol{at}\ p} \to I \qquad \frac{\Gamma;\Delta \vdash^P \phi \to \psi\ \boldsymbol{at}\ p \quad \Gamma;\Delta \vdash^P \phi\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash \psi\ \boldsymbol{at}\ p} \to E$$

$$\frac{\Gamma;\Delta \vdash^P \phi\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash^P \phi@p\ \boldsymbol{at}\ p'}\ @I \qquad \frac{\Gamma;\Delta \vdash^P \phi@p\ \boldsymbol{at}\ p'}{\Gamma;\Delta \vdash^P \phi\ \boldsymbol{at}\ p}\ @E$$

$$\frac{\Gamma;\Delta \vdash^P \phi\ \boldsymbol{at}\ p}{\Gamma;\Delta \vdash^P \Diamond\phi\ \boldsymbol{at}\ p'}\ \Diamond I \qquad \frac{\Gamma;\Delta \vdash^P \Diamond\phi\ \boldsymbol{at}\ p' \quad \Gamma;\Delta,\phi\ \boldsymbol{at}\ q \vdash^{P+q} \psi\ \boldsymbol{at}\ p''}{\Gamma;\Delta \vdash^P \psi\ \boldsymbol{at}\ p''}\ \Diamond E$$

$$\frac{\Gamma;\Delta \vdash^{P+q} \phi\ \boldsymbol{at}\ q}{\Gamma;\Delta \vdash^P \Box\phi\ \boldsymbol{at}\ p}\ \Box I \qquad \frac{\Gamma;\Delta \vdash^P \Box\phi\ \boldsymbol{at}\ p \quad \Gamma,\phi;\Delta \vdash^P \psi\ \boldsymbol{at}\ p'}{\Gamma;\Delta \vdash^P \psi\ \boldsymbol{at}\ p'}\ \Box E$$

FIGURE 1. Natural deduction.

analogy, however, has to be taken carefully. For example, if $\Gamma;\Delta \vdash^P \Diamond\psi\ \boldsymbol{at}\ p$, then we can show using the rules of the logic that $\Gamma;\Delta \vdash^P \Box\Diamond\psi\ \boldsymbol{at}\ p$. In other words, if a formula $\psi$ is true in some unspecified place, then every place can deduce that there is some place where $\psi$ is true.

Also note that, as stated, the rule $\bot E$ has a 'local' flavour: from $\bot\ \boldsymbol{at}\ p$, we can infer any other property in the same place, $p$. However, the rule has a 'global' consequence. If we have $\bot\ \boldsymbol{at}\ p$, then we can infer $\bot@q\ \boldsymbol{at}\ p$. Using $@E$, we can then infer $\bot\ \boldsymbol{at}\ q$. Hence, if a set of assumptions makes a place inconsistent, then it will make all places inconsistent.

As we shall see in §2.1, the Kripke semantics of this logic would be similar to the one given for intuitionistic system *IS5* [23, 29, 35]. Hence this logic can be seen as an instance of *Hybrid IS5* [7]. Before we proceed to define the Kripke semantics, we illsutrate our derivation system by a couple of examples. First example will demonstrate the use of rule $\Diamond I$, while the second example will demonstrate the use of $\Box E$.

**Example 1** Let $p, p' \in P$, $\psi$ be the formula $(\mathsf{doc}_2 \wedge (\mathsf{doc}_2 \to \mathsf{doc}_2@p))\ \boldsymbol{at}\ p'$. Let $\Delta \stackrel{\text{def}}{=} \Diamond\psi$. Pick $q \notin P$ and let $\Delta' \stackrel{\text{def}}{=} \Diamond\psi; \Delta\psi \vdash^P$

$$\dfrac{\dfrac{}{;\Delta\ \square^{P}\ \blacklozenge\psi\ \textbf{\textit{at}}\ p^{\square}}\ L \qquad \begin{array}{c}\vdots\ \pi\\ ;\Delta^{\square}\ \square^{P+q}\ \mathrm{doc}_2\ \textbf{\textit{at}}\ p\end{array}}{;\Delta\ \square^{P}\ \mathrm{doc}_2\ \textbf{\textit{at}}\ p}\ \blacklozenge E$$

where π is the derivation:

$$\dfrac{\dfrac{}{;\Delta^{\square}\ \square^{P+q}\ \mathrm{doc}_2\ \square\ (\mathrm{doc}_2 \rightarrow \mathrm{doc}_2)\ \textbf{\textit{at}}\ q}\ L}{;\Delta^{\square}\ \square^{P+q}\ \mathrm{doc}_2\ \textbf{\textit{at}}\ q}\ \square E \qquad \dfrac{}{;\Delta^{\square}\ \square^{P+q}\ \mathrm{doc}_2\ \square\ (\mathrm{doc}_2 \rightarrow \mathrm{doc}_2)\ \textbf{\textit{at}}\ q}\ L$$

The Kripke models that we shall define now are similar to those defined for the intuitionistic modal system *IS5* [11, 34, 23, 26, 7, 35]. In the definition, $K$ is the set of Kripke states, and its elements are denoted by $k, l, \ldots$ The relation $\leq$ is the partial order on the set of states.

**Definition 3 (Kripke Model)** A quadruple $\mathcal{K} = (K, \leq, \{P_k\}_{k \in K}, \{I_k\}_{k \in K})$ is a *Kripke model* if

- $K$ is a (non empty) set;
- $\leq$ is a partial order on $K$;
- $P_k$ is a *non-empty* set of places for all $k \in K$;
- $P_k \subseteq P_l$ if $k \leq l$;
- $I_k : Atoms \to Pow(P_k)$ is such that $I_k(A) \subseteq I_l(A)$ for all $k \leq l$.

Let $Pls = \bigcup_{k \in K} P_k$. We shall say that $Pls$ is the set of places of $\mathcal{K}$.

The definition tells only how resources, i.e. atoms, are distributed in the system. To give semantics to the whole set of formulae $Frm(Pls)$, we need to extend $I_k$. The interpretation of a formula depends on its composite parts, and if it is valid in a place in a given state, then it remains valid at the same place in all larger states. For example, the formula $\phi \supset \psi$ is valid in a state $k$ at place $p \in P_k$, if both $\phi$ and $\psi$ are true at place $p$ in all states $l \geq k$.

The introduction of places in the model allows the interpretation of the spatial modalities of the logic. Formula $\phi@p$ is satisfied at a place in a state $k$, if it is true at $p$ in all states $l \geq k$; $\blacklozenge\phi$ and $\Box\phi$ are satisfied at a place in state $k$, if $\phi$ is true respectively at some or at every place in all states $l \geq k$.

We extend now the interpretation of atoms to interpretation of formulae by using induction on the structure of the formulae. The interpretation of formulae is similar to that used for modal intuitionistic logic [11, 34, 23, 26, 7, 35].

**Definition 4 (Semantics)** Let $\mathcal{K} = (K, \leq, \{P_k\}_{k \in K}, \{I_k\}_{k \in K})$ be a Kripke model with set of places $Pls$. Given $k \in K$, Consider now the distributed database described before. We can express the same properties inferred in §2 by using a Kripke model. Fix a Kripke state $k$. The assumption that the two parts, $\mathsf{doc}_1, \mathsf{doc}_2$, can be combined in $p$ in a state $k$ to give the document $\mathsf{doc}$ can be expressed as $(k, p) \models (\mathsf{doc}_1 \wedge \mathsf{doc}_2) \to \mathsf{doc}$. If the resources $\mathsf{doc}_1$ and $\mathsf{doc}_2$ are assigned to the place $p$, i.e., $(k, p) \models \mathsf{doc}_1$ and $(k, p) \models \mathsf{doc}_2$, then, since $(k, p) \models \mathsf{doc}_1 \wedge \mathsf{doc}_2$, it follows that $(k, p) \models \mathsf{doc}$.

Let us consider a slightly more complex situation. Suppose that $k \models \Diamond(\mathsf{doc}_2 \wedge (\mathsf{doc}_2 \to \Box\mathsf{doc}_2))$ ***at*** $p'$. According to the semantics of $\Diamond$, there is some place $r$ such that $(k, r) \models \mathsf{doc}_2 \wedge (\mathsf{doc}_2 \to \Box\mathsf{doc}_2)$. The semantics of $\wedge$ tells us that $(k, r) \models \mathsf{doc}_2$ and $(k, r) \models (\mathsf{doc}_2 \to \Box\mathsf{doc}_2)$. Since $(k, r) \models \mathsf{doc}_2$, we know from the semantics of $\to$ that $(k, r) \models \Box\mathsf{doc}_2$, and from the semantics of $\Box$ that $(k, p) \models \mathsf{doc}_2$. Therefore, if $\mathsf{doc}_1$ is placed at $p$ in the state $k$, then the whole document $\mathsf{doc}$ would become available at place $p$ in state $k$.

To give semantics to the judgements of the logic, we need to extend the definition of forcing relation to judgements. We begin by extending the definition to contexts.

**Definition 6 (Forcing on Contexts)** Let $\mathcal{K} = (K, \leq, \{P_k\}_{k \in K}, \{I_k\}_{k \in K})$ be a Kripke model. Given a state $k$ in $K$, a finite set of pure formulae $\Gamma$, and a finite set of sentences $\Delta$ such that $\mathsf{PL}(\Gamma; \Delta) \subseteq P_k$; we say that $k$ forces the context $\Gamma; \Delta$ (and we write $k \models \Gamma; \Delta$) if

1. for every $\phi \in \Gamma$ and every $p \in P_k$: $(k, p) \models \Box\phi$;
2. for every $\psi$ ***at*** $q \in \Delta$: $(k, q) \models \psi$.

Finally, we extend the definition of forcing to judgements.

**Definition 7 (Satisfaction for a Judgment)** Let $\mathcal{K} = (K, \leq, \{P_k\}_{k \in K}, \{I_k\}_{k \in K})$ be a Kripke model. The judgement $\Gamma; \Delta \vdash^P \mu$ **at** $p$ *is said to be valid in* $\mathcal{K}$ if

- $\mathsf{PL}(\Gamma) \cup \mathsf{PL}(\Delta) \cup \mathsf{PL}(\mu) \cup \{p\} \subseteq P$;
- for every $k \in K$ such that $P \subseteq P_k$, if $k \models \Gamma; \Delta$ then $(k, p) \models \mu$.

Moreover, we say that $\Gamma; \Delta \vdash^P \mu$ **at** $p$ *is valid* (and we write $\Gamma; \Delta \models \mu$ ***at*** $p$) if it is valid in every Kripke model.

Although, it is possible to obtain soundness and completeness of Kripke semantics directly, we shall not do so in this paper. Instead, they will be derived as corollaries. Soundness will follow from the soundness of birelational semantics and encoding of Kripke models into birelational models. Completeness will emerge as a corollary in the proof of construction of finite counter-model.

## 3 Birelational Models

One other semantics given for modal intuitionistic logics in literature is birelational semantics [11, 34, 26, 35]. As in the case of intuitionistic modal logics [24, 35], birelational semantics for our logic enjoys the finite model property, while Kripke semantics does not.

Birelational models, like Kripke models, have a set of partially ordered states. The partially ordered states will be called *worlds*, and we use $u, v, w, \ldots$ to range over them. Formulae will be validated in worlds, and if a formula is validated in a world, then it will be validated in all larger worlds. To validate atoms we have the interpretation $I$, which maps atoms into a subset of worlds. If $I$ maps an atom into a world, then it will map the atom in all larger worlds.

In addition to the partial order, however, there is also a second binary relation on the set of states which is called *reachability* or *accessibility* relation. Intuitively, $uRw$ means that $w$ will be reachable from $u$. As our logic is a hybridisation for *IS5*, the relation $R$ will be an equivalence relation. The relation $R$ will also satisfy a technical requirement, the *reachability condition*, that is necessary to ensure monotonicity and soundness of logic evaluation.

Unlike the Kripke semantics, the states will not have a set of places associated to them. Instead, there is a *partial* function, *Eval*, which maps a world to a *single* place. In a sense which we will make precise in §3.2, a world in a birelational model corresponds to a place in a specific Kripke state. As we shall see later, the partiality of the function *Eval* is crucial in the proof of the finite model property. In the case *Eval*($w$) is defined and is $p$, we shall say that $w$ *evaluates* to $p$. $w$ *evaluates* to $p$.

**Proposition 10 (Monotonicity)** Let $\mathsf{W}_{Pls}$ be a birelational model on *Pls*. The relation $\models$ preserves the partial order in $W$, namely, for every world $w$ in $W$ and $\phi \in Frm(Pls)$, if $v \geq w$ then $w \models \phi$ implies $v \models \phi$.

**Proof** The proof is straightforward, and proceeds by induction on the structure of formulae. Here, we just consider the induction step in which $\phi$ is of the form $\phi_1@p$. Suppose that $w \models \phi_1@p$. Then there is a $w'$ such that $w\,R\,w'$, $w' \downarrow p$ and $w' \models \phi_1$.

Consider now $v \geq w$. Since $w\,R\,w'$, by the reachability condition we obtain that there is a world $v'$ such that $v\,R\,v'$ and $v' \geq w'$. As $w' \models \phi_1$, by induction hypothesis we obtain $v' \models \phi_1$. Now, as $v' \geq w'$ and $w' \downarrow p$, we get $v' \downarrow p$ by coherence property. Finally, as $v\,R\,v'$, we get $v \models \phi_1@p$ by definition. □

**Example 11** Consider the birelational model $\mathsf{W}_{exam}$ with two worlds, say $w_1$ and $w_2$. We take $w_1 \leq w_2$, and both worlds are reachable from each other. The world $w_2$ evaluates to $p$, while the evaluation of $w_1$ is undefined. Let $A$ be an atom. We define $I(A)$ to be the singleton $\{w_2\}$. For any formula $\phi$, we abbreviate $\phi \to \bot$ as $\neg\phi$.

Consider the pure formula $\neg A$. Now, by definition, $w_2 \models A$ and therefore $w_2 \not\models \neg A$. Also, as $w_1 \leq w_2$, we get $w_1 \not\models \neg A$. This means that $w_2 \models \neg\neg A$, and $w_1 \models \neg\neg A$. Hence, we get $w_1, w_2 \models \Box\neg\neg A$.

On the other hand, consider the formula $\neg\neg\Box A$. We have by definition that $w_1 \not\models A$. As $w_1$ is reachable from both $w_1$ and $w_2$, we deduce that $w_1, w_2 \not\models \Box A$. Using the semantics of $\to$, we get that $w_1, w_2 \not\models \neg\neg\Box A$.

We now extend the semantics to the judgements of the logic. We begin by extending the semantics to contexts.

**Definition 12 (Bi-forcing on Contexts)** Let $\mathsf{W}_{Pls} = (W, \leq, R, I, Eval)$ be a birelational model on *Pls*. Given a finite set of pure formulae $\Gamma$, and a finite set of sentences $\Delta$, such that $\mathsf{PL}(\Gamma; \Delta) \subseteq Pls$; we say that $w \in W$ forces the context $\Gamma; \Delta$ (and we write $w \models \Gamma; \Delta$) if

1. for every $\phi \in \Gamma$: $w \models \Box\phi$, and
2. for every $\psi$ ***at*** $q \in \Delta$: $w \models \psi@q$.

In order to extend the semantics to judgements, we need one more definition. We say that a place $p$ is reachable from a world $v$, if there is a world which evaluates to $p$ and is reachable from $v$. The set of all places reachable from a world $v$ will be denoted by $Reach(v)$. More formally,

$$Reach(v) \stackrel{\text{def}}{=} \{p : w\downarrow p \text{ for some } w \; \square \quad Reach$$

$;\vdash^{\{p\}} \Box\neg\neg A$ ***at*** $p$ is bi-valid in the model $\mathsf{W}_{exam}$, while the judgement $;\Box\neg\neg A$ ***at*** $p \vdash^{\{p\}} \neg\neg\Box A$ ***at*** $p$ is not bi-valid in $\mathsf{W}_{exam}$. In fact, we will later on show that the judgement $;\Box\neg\neg A$ ***at*** $p \vdash^{\{p\}} \neg\neg\Box A$ ***at*** $p$ is valid in every finite Kripke model. Therefore, this example, adapted from [24, 35], will demonstrate that the finite model property does not hold in the case of Kripke semantics.

### 3.1 Soundness

The proof of soundness of birelational models has several subtleties, that arise as a consequence of the inference rules for the introduction of $\Box$ ($\Box$ I), and elimination of $\Diamond$ ($\Diamond$ E). Let us illustrate this for the case of $\Box I$. Recall the inference rule of $\Box I$ from Fig. 1:

$$\frac{\Gamma;\Delta \vdash^{P+q} \phi \textbf{\textit{ at }} q}{\Gamma;\Delta \vdash^{P} \Box\phi \textbf{\textit{ at }} p}\ \Box I$$

To show the soundness of this rule, we must show that the judgement $\Gamma;\Delta \vdash^{P} \Box\phi$ ***at*** $p$ is bi-valid whenever the judgement $\Gamma;\Delta \vdash^{P+q} \phi$ ***at*** $q$ is bi-valid. Now, to show that the judgement $\Gamma;\Delta \vdash^{P} \Box\phi$ ***at*** $p$ is bi-valid, we must consider an arbitrary world, say $w$, in an arbitrary birelational model, say $\mathsf{W}_{Pls}$, such that $P \subseteq Reach(w)$ and $w \models \Gamma;\Delta$. We need to prove that $w \models \Box\phi@p$ also. For this, we need to show that for any world $v$ in $\mathsf{W}_{Pls}$ such that $w \leq w' R\, v$ for some $w'$, it is the case that $v \models \phi$. Pick one such $v$ and fix it.

Please note that without loss of generality, we can assume that $Pls$ does not contain $q$ (otherwise, we can always rename $q$ in the model). To use the hypothesis that $\Gamma;\Delta \vdash^{P+q} \phi$ ***at*** $q$ is bi-valid, we must consider a modification of $\mathsf{W}_{Pls}$. One strategy, that is adopted in the case of Kripke semantics [7], is to add new worlds $v'_q$, one for each world $v' \geq v$. The new worlds Γ [illegible]

$(q, v)$ satisfies $\phi$. As mentioned above, this is equivalent to saying that $v$ satisfies $\phi$.

We are ready to carry out this proof formally. We begin by constructing the $q$-extension, and showing that this is a birelational model.

**Lemma 16 ($q$-Extension)** Let $\mathcal{W}_{Pls} = (W, \leq, R, I, Eval)$ be a birelational model on $Pls$. Given a new place $q \notin Pls$, we define the $q$-extension $\mathcal{W}[q]_{Pls'}$ to be the quintuple $(W', \leq', R', I', Eval')$, where

1. $Pls' \stackrel{\text{def}}{=} Pls \cup \{q\}$.
2. $W' \stackrel{\text{def}}{=} R \cup (\{q\} \times W)$.
3. $\leq' \subseteq W' \times W'$ is defined as:
   - $(w', w) \leq' (v', v)$ if and only if $w' \leq v'$ and $w \leq v$,
   - $(q, w) \leq' (q, v)$ if and only if $w \leq v$;
4. $R' \subseteq W' \times W'$ is defined as:
   - $(w', w)\, R'\, (v', w)$,
   - $(w', w)\, R'\, (q, w)$,
   - $(q, w)\, R'\, (w', w)$, and
   - $(q, w)\, R'\, (q, w)$.
5. $I' : Atoms \to Pow(W')$ is defined as:
   - $I'(A) \stackrel{\text{def}}{=} \{ (w', w) \mid w' \in I(A),\ w' R\, w \} \cup \{ (q, w) \mid w \in I(A) \}$;
6. $Eval' : W' \to Pls'$ is defined as
   - $Eval'((w', w)) \stackrel{\text{def}}{=} Eval(w')$ for every $(w', w) \in R$,[1]
   - $Eval'((q, w)) \stackrel{\text{def}}{=} q$ for every $w \in W$.

The $q$-extension is a birelational model.

**Proof** We need to show the five properties of Definition 8.

1. Clearly $W'$ is a non empty set if $W$ is.
2. Since $\leq$ is a partial order, then $\leq'$ is a partial order too.
3. The relation $R'$ is an equivalence by definition. We show that $R'$ satisfies the reachability condition by cases. There are four possible cases.

Case a. Assume that $(v', v) \geq' (w', w)\, R'\, (w'', w)$.
The hypothesis says that $v \geq w$, $v' \geq w'$, $v' R\, v$, $w' R\, w$ and $w'' R\, w$. Since $R$ is an equivalence, we get $v' \geq w' R\, w''$. Using the reachability condition for $R$, there exists $v'' \in W$ such that $v' R\, v'' \geq w''$. Hence, we conclude $(v', v)\, R'\, (v'', v) \geq (w', w)$.

Case b. Assume that $(q, v) \geq' (q, w)\, R'\, (w', w)$.
This means that $v \geq w$ and $w\, R\, w'$. By the reachability condition for $R$, there is a $v'$ such that $v\, R\, v' \geq w'$, and we conclude $(q, v)\, R'\, (v', v) \geq' (w', w)$.

Case c. Assume that $(v', v) \geq' (w', w)\, R'\, (q, w)$.
This means $v \geq w$, and we conclude $(v', v)\, R'\, (q, v) \geq' (q, w)$.

Case d. Assume that $(q, v) \geq' (q, w)\, R'\, (q, w)$.
We have $v \geq w$, and we conclude $(q, v)\, R'\, (q, v) \geq' (q, w)$.

4. To check monotonicity for $I'$, we consider two cases:

[1] In the equality, the left hand side is defined only if the right hand side is.

Case a. Assume that $(w', w) \in I'(A)$.
This means that $w' \in I(A)$. If $(v', v) \geq' (w', w)$, then $v' \geq w'$. By the monotonicity of $I$, we get $v' \in I(A)$. Hence $(v', v) \in I'(A)$.

Case b. Assume that $(q, w) \in I(A)$.
This means that $w \in I(A)$. If $(q, v) \geq' (q, w)$, then $v \geq w$. By the monotonicity of $I$, we get $v \in I(A)$. Hence $(q, v) \in I'(A)$.

5. According to the definition, $Eval'$ is a partial function. We need to verify the two properties required for a birelational model.

   *Coherence*. We have to show that if a world in the new model evaluates to some place, then all the higher worlds evaluate to the same place. There are two possible cases.

   Case a. Assume that $(v', v) \geq' (w', w)$, and $(w', w) \downarrow p$
   We get by definition, $v' \geq w'$ and $w' \downarrow p$. By coherence on the model $\mathcal{W}_{Pls}$, we get $v' \downarrow p$. Hence $(v', v) \downarrow p$.

   Case b. Assume that $(q, v) \geq' (q, w)$.
   We have by definition, $(q, v) \downarrow q$ and $(q, w) \downarrow q$.

   *Uniqueness*. We have to show that two different worlds reachable from each other cannot evaluate to the same place. As $(q, v)$ always evaluates to $q$, two worlds $(w, v)$ and $(q, w)$ cannot evaluate to the same place. There are two other possible cases.

   Case a. Suppose $(v', v)\, R'\, (w', w)$, $(w', w) \downarrow p$ and $(v', v) \downarrow p$.
   We have by definition $v' R\, v$, $w' R\, w$, $v = w$, $w' \downarrow p$ and $v' \downarrow p$. Since $R$ is an equivalence and $v = w$, we get $v' R\, w'$. By uniqueness on $\mathcal{W}_{Pls}$, we get $v' = w'$. Therefore $(v', v) =' (w', w)$

   Case b. Suppose that $(q, v)\, R'\, (q, w)$, $(q, w) \downarrow q$ and $(q, v) \downarrow q$.
   We have by definition $v = w$, and hence $(q, v) = (q, w)$. ∎

We will now show that if a pure formula, say $\psi$, does not mention $q$, then $(w', w)$ satisfies $\psi$ only if $w'$ does. Furthermore, $(q, w)$ satisfies $\psi$ only if $w$ does.

**Lemma 17 ($\mathcal{W}\langle u, q\rangle_{Pls'}$ is conservative)** Let $\mathcal{W}_{Pls} = (W, \leq, R, I, Eval)$ be a birelational model, and let $\mathcal{W}\langle q\rangle_{Pls'} = (W', \leq', R', I', Eval')$ be its $q$-extension. Let $\models$ and $\models'$ extend the interpretation of atoms in $\mathcal{W}_{Pls}$ and $\mathcal{W}\langle q\rangle_{Pls'}$ respectively. For every $\phi \in Frm(Pls)$ and $w \in W$, it holds

1. for every $w' R\, w$, $(w', w) \models' \phi$ if and only if $w' \models \phi$; and
2. $(q, w) \models' \phi$ if and only if $w \models \phi$.

**Proof** Prove both the points simultaneously by induction on the structure of formulae in $Frm(Pls)$.

*Base of induction.* The two points are verified on atoms, on $\bot$, and on $\top$ by definition.

*Induction hypothesis.* We consider a formula $\phi \in Frm(Pls)$, and assume that the two points hold for all sub-formulae $\phi_i$ of $\phi$. In particular, we assume that *for every $w \in W$:*

1. *for every $w' R\, w$, $(w', w) \models' \phi_i$ if and only if $w' \models \phi_i$; and*
2. *$(q, w) \models' \phi_i$ if and only if $w \models \phi_i$.*

We shall prove the lemma only for the modal connectives and for the logical connective $\rightarrow$. The other cases can be treated similarly. We shall also only consider point *1*, as the treatment of point *2* is analogous. We pick $w \in W$ and $w' R\, w$, and fix them.

- *Case* $\phi = \phi_1 \rightarrow \phi_2$. Suppose $(w', w) \models' \phi_1 \rightarrow \phi_2$. Then

$$\text{for every } (v', v) \geq' (w', w), \text{ we have } (v', v) \models' \phi_1 \text{ implies } (v', v) \models' \phi_2. \tag{2}$$

  We need to show that $w' \models \phi$. Pick $v' \geq w'$ such that $v' \models \phi_1$, and fix it. It suffices to show that $v' \models \phi_2$.

We have $v' \geq w' R w$. By the reachability condition, there exists $v \in W$ such that $v' R v \geq w$. Hence, $(v', v) \geq' (w', w)$.

The induction hypothesis says that $(v', v) \models' \phi_1$. We have $(v', v) \models' \phi_2$ by (2) above. Hence $v' \models \phi_2$, by applying induction hypothesis one more time.

For the other direction, assume that $w' \models \phi_1 \to \phi_2$. Then

$$\text{for every } v' \geq w', \text{ we have } v' \models \phi_1 \text{ implies } v' \models \phi_2. \tag{3}$$

Now consider $(v', v) \geq' (w', w)$, and assume $(v', v) \models' \phi_1$. From $(v', v) \geq' (w', w)$, we have $v' \geq w'$. From $(v', v) \models' \phi_1$

**Proposition 18 (Forcing Propagation)** Let $\mathsf{W}_{Pls} = (W, \leq, R, V, Eval)$ be a birelational model on *Pls*. Let Γ be a finite set of pure formulae, Δ be a finite set of sentences Δ, and *w* be a world in *W* such that $w \models \Gamma; \Delta$. Then

ow1(2.)]TJ,963 Tf 7.747 0 Td[(;2]9J/F73 9wd[(,)-250(and)]TJ -1

1. $v \models \Gamma; \Delta$ for every $v\,R\,w$, and
2. $v \models \Gamma; \Delta$ for every $v \geq w$.

**Proof** The second part of the proposition is an easy consequence of monotonicity of the logic. For the first part, pick $v\,R\,w$ and fix it. We need to show that if ψ is a formula in Γ then $v \models \Box\psi$, and that if ϕ ***at*** *p* is a sentence in Δ then $v \models \phi@p$.

Now, if $\psi \in \Gamma$, then we have that $w \models \Box\psi$. Let $v', v''$ be two worlds such that $v''\,R\,v' \geq v$. We will show that

2@ $R\,v$ □ψ. Let .403

fix it. By Proposition 18, $w^{\square} \models \Gamma; \Delta$. We shall show that $w^{\square} \models \square\phi$, and we will be done.

In order to show that $w^{\square} \models \square\phi$, we have to show that $v^{\square} \models \phi$

Assume that $(k', p') \geq' (k, p)\, R'(l, q)$. Then it must be the case that $k' \geq k$, $k = l$ and $q \in P_l$. Since $k = l$, we get $q \in P_k$. Furthermore, as $k' \geq k$, we have $P_k \subseteq P_{k'}$. Therefore $q \in P_{k'}$.

Consider the world $(k', q)$. We get $(k', p')\, R'(k', q) \geq' (k, q)$ by definition. $\square$

The encoding preserves the forcing relation:

**Proposition 21 (Forcing Preservation)** Let $\mathsf{K} = (K, \leq, \{P_k\}_{k \in K}, \{I_k\}_{k \in K})$ be a Kripke model with set of places *Pls*. Let $\mathsf{W}^{\mathsf{K}}_{Pls} = (W', \leq', R', I', Eval')$ be the $\mathsf{K}$-birelational model. Let $\models_{\mathsf{K}}$ and $\models_{\mathsf{W}}$ extend the interpretation of atoms in $\mathsf{K}$ and $\mathsf{W}^{\mathsf{K}}_{Pls}$ respectively. For every $\phi \in Frm(Pls)$, $k \in K$, and $p \in P_k$, we have:

$$(k, p) \models_{\mathsf{K}} \phi \text{ if and only if } (k, p) \models_{\mathsf{W}} \phi.$$

**Proof** We proceed by induction on the formula $\phi \in Frm(Pls)$. The statement of the proposition is easily verified on $\bot$, $\top$ and on atoms.

*Induction hypothesis.* We consider a formula $\phi \in Frm(Pls)$, and;

and order are essentially orthogonal. Hence, the reverse encoding will fail to preserve the forcing relation.

This is no accident, and as we have pointed out before, partiality of the evaluation in birelational models is essential for the proof of the finite model property. This was illustrated by the “finite model” $\mathsf{W}_{exam}$ in Ex. 11. In $\mathsf{W}_{exam}$, it is the case that $w_1 \leq w_2$, $w_1 \, R \, w_2$, $w_1\uparrow$ and $w_2\downarrow p$. As discussed there, this model allows us to refute the judgement $; \Box\neg\neg A$ ***at*** $p \vdash^{\{p\}} \neg\neg\Box A$ ***at*** $p$. As we will see later, the judgement will be valid in every finite Kripke model.

We shall now use the encoding and soundness of logic with respect to birelational models to show soundness of Kripke semantics.

**Corollary 22 (Soundness)** If $\Gamma; \Delta \vdash^P \mu$ ***at*** $p$ is derivable in the logic, then it is valid in every Kripke model.

**Proof** Suppose that the judgement $\Gamma; \Delta \vdash^P \mu$ ***at*** $p$ is derivable. Then it must be the case that $\mathsf{PL}(\Gamma) \cup \mathsf{PL}(\Delta) \cup \mathsf{PL}(\mu) \cup \{p\} \subseteq P$. Let $\mathsf{K} = (K, \leq, \{P_k\}_{k \in K}, \{I_k\}_{k \in K})$ be a Kripke model with set of places *Pls*. Let $\models_{\mathsf{K}}$ extend the interpretation of atoms to formulae on this Kripke model. Let $k$ be a Kripke state of this model such that $P \subseteq P_k$ and $k \models_{\mathsf{K}} \Gamma; \Delta$. We need to show that $(k, p) \models_{\mathsf{K}} \mu$.

Consider the encoding of the Kripke model $\mathsf{K}$ into a birelational model. Let $\mathsf{W}^{\mathsf{K}}_{Pls} = (W^{\star}, \leq^{\star}, R^{\star}, I^{\star}, Eval^{\star})$ be the $\mathsf{K}$-birelational model, and consider the world $(k, p) \in W^{\star}$. If $\models_{\mathsf{W}}$ is the extension of interpretation of atoms in this model, we claim that $(k, p) \models_{\mathsf{W}} \Gamma; \Delta$.

If $\psi \in \Delta$ then as $k \models_{\mathsf{K}} \Gamma; \Delta$, we get by definition $(k, p) \models_{\mathsf{K}} \Box\psi$. By Proposition 21, we get that $(k, p) \models_{\mathsf{W}} \Box\psi$.

If $\psi$ ***at*** $q \in \Gamma$, then we have by definition $(k, q) \models_{\mathsf{K}} \psi$. By Proposition 21, we get that $(k, q) \models_{\mathsf{W}} \psi$. Now, by construction $(k, p) \, R^{\star} (k, q)$, and hence we get $(k, p) \models_{\mathsf{W}} \psi@q$.

Therefore, we get that $(k, p) \models_{\mathsf{W}} \Gamma; \Delta$. As the logic is sound over birelational models, we get $(k, p) \models_{\mathsf{W}} \mu@p$. This implies that $(k, p) \models_{\mathsf{K}} \mu@p$, by Proposition 21 once again. Finally, this is the same as $(k, p) \models_{\mathsf{K}} \mu$, by definition, and we have done. □

## 4 Bounded contexts and Completeness

In this section, we shall prove completeness of the logic with respect to both Kripke and birelational semantics. The proof will follow a modification of standard proofs of completeness of intuitionistic logics[19, 35, 7, 36], and we will construct a particular Kripke model: the *canonical bounded Kripke model*. The reason for the term “bounded” shall become clear later on. We will prove that a judgement $\Gamma; \Delta \vdash^P \mu$ ***at*** $p$ is valid in the canonical bounded model if and only if it is derivable in the logic. Then we will use the encoding of the Kripke models into birelational models (see §3.2), which will allow us to prove completeness of birelational models. The resulting model will be used to prove the finite model property in §5.3. The construction of the model is adapted from [35].

We also point out that we shall prove the completeness results in the case where $P$ is finite. This is not a serious restriction for completeness, and the result can be extended to judgements where $P$ is infinite. The real advantage of using a finite set of places is that it will assist in the proof of finite model property as we will see in §5.

We begin by defining sub-formulae of a pure formula. A *sub-formula* of $\varphi$, a pure formula

**Definition 23 (Bounded Contexts)** Given a finite set of places $P$ and a finite set of pure formulae $\Theta \subseteq Frm(P)$, a pair $(Q, \Delta)$ is a $(P, \Theta)-$*bounded context* if

- $Q$ is a finite set of places that contains $P$, *i.e.*, $P \subseteq Q$; and
- $\Delta$ is a finite set of sentences of the form $\phi$ ***at*** $q$, where $\phi \in \Theta^{\lhd}$ and $q \in Q$.

The bounded contexts will be used as Kripke states in the canonical model. However, we will need particular kinds of bounded contexts.

**Definition 24 (Prime Bounded Contexts)** Let $\Theta, \Gamma \subseteq Frm(P)$ be two finite sets of pure formulae on the finite set of places $P$. A $(P, \Theta)-$bounded context $(Q, \Delta)$ is said to be $\Gamma-$*prime* if

- $\Gamma; \Delta \vdash^{Q} \phi$ ***at*** $q$ for $\phi \in \Theta^{\lhd}$ and $q \in Q$, implies that $\phi$ ***at*** $q \in \Delta$ ($\Theta$-deductive closure);
- $\Gamma; \Delta \nvdash^{Q} \bot$ ***at*** $q$ for every $q \in Q$ (Consistency);
- $\Gamma; \Delta \vdash^{Q} \phi \vee \psi$ ***at*** $q$ for $\phi \vee \psi \in \Theta^{\lhd}$ and $q \in Q$, implies that either $\phi$ ***at*** $q \in \Delta$ or $\psi$ ***at*** $q \in \Delta$ ($\Theta$-disjunction property); and
- $\Gamma; \Delta \vdash^{Q} \blacklozenge\phi$ ***at*** $q$ for $\blacklozenge\phi \in \Theta^{\lhd}$ and $q \in Q$, implies that there exists $q' \in Q$ such that $\phi$ ***at*** $q' \in \Delta$ ($\Theta$-diamond property).

As an example, let $A$ be an atom. Let $P = \{p\}$, $\Theta = \{A@p\}$ and $Q = \{p, q\}$. Consider the following sets of sentences:

- $\Delta_1 = \{A$ **at** $p,\ A$ **at** $q,\ A@p$ **at** $p\}$;
- $\Delta_2 = \{A$ **at** $p,\ A$ **at** $q,\ A@p$ **at** $p,\ A@p$ **at** $q\}$; and
- $\Delta_3 = \{A$ **at** $p,\ A$ **at** $q,\ A@p$ **at** $p,\ A@p$ **at** $q,\ \blacklozenge A$ **at** $q\}$.

Clearly, we have that $P \subseteq Q$. If $\psi$ ***at*** $r$ is a sentence in $\Delta_1$ or $\Delta_2$, then $\psi$ is a sub-formula of $\Theta$ and $r \in Q$. Therefore, $(Q, \Delta_1)$ and $(Q, \Delta_2)$ are $(P, \Theta)-$ bounded contexts. On the other hand, $(Q, \Delta_3)$ is not a $(P, \Theta)-$bounded context as $\blacklozenge A$ is not a sub-formula of $A@p$.

If we let $\Gamma$ to be the list $\{A\}$, then it follows easily that $\Gamma; \Delta_1 \vdash^{Q} A$ ***at*** $p$. Using the inference rule of introduction of @, we get $\Gamma; \Delta_1 \vdash^{Q} A@p$ ***at*** $q$. However, we have that $A@p$ ***at*** $q \notin \Delta_1$. Therefore, $(Q, \Delta_1)$ is not $\Gamma-$prime. On the other hand, $(Q, \Delta_2)$ is $\Gamma-$prime.

The set $\Delta^*$ required in the lemma would be a subset of $\Sigma$, and the set $Q^*$ would be a subset of $Q \cup Q_\blacklozenge$. These sets would be obtained by a series of extensions $\Delta_n$, $Q_n$ which will satisfy certain properties:

*Property 1* For every $n \geq 0$

1. $Q_n \subseteq Q \cup Q_\blacklozenge$, and $\Delta_n \subseteq \Sigma$;
2. $Q_n \subseteq Q_{n+1}$, $\Delta_n \subseteq \Delta_{n+1}$;
3. $(Q_n, \Delta_n)$ is $(P, \Theta)$-bounded context; and
4. $\Gamma; \Sigma_n \nvdash^{Q_n} \psi$ ***at*** $q$.

The series is constructed inductively. In the induction, at an odd step we will create a witness for a formula of the type $\blacklozenge\phi$. At an even step we deal with disjunction property. We shall also construct two sets:

- $\mathit{treated}_n^\blacklozenge$, that will be the set of the formulae $\blacklozenge\phi \in \Theta^*$ for which we have already created a witness.
- $\mathit{treated}_n^\vee$, that will be the set of the formulae $\psi_1 \vee \psi_2$ ***at*** $q \in \Sigma$ which satisfy the disjunction property.

We pick an enumeration of $\Theta^*$, and fix it. We start off by defining $\mathit{treated}_0^\blacklozenge = \emptyset$, $\mathit{treated}_0^\vee = \emptyset$, $Q_0 = Q$, and $\Delta_0 = \Delta$. It is clear from the hypothesis of the lemma that $Q_0$ and $P_0$ satisfy the four points of Property1.

Then we proceed inductively, and assume that $Q_n, \Delta_n$ $(n \geq 0)$ have been constructed satisfying Property 1. In step $n + 1$, we consider two cases:

1. If $n + 1$ is odd, then pick the first formula $\psi_1 \vee \psi_2 \in \Theta^*$ in the enumeration of $\Theta^*$, such that
   - $\Gamma; \Delta_n \vdash^{Q_n} \psi_1 \vee \psi_2$ ***at*** $r$, for some $r \in Q_n$;

This contradicts the hypothesis on $Q_n, \Delta_n$. Hence $\Gamma; \Delta_{n+1} \nvdash^{Q_{n+1}} \psi$ ***at*** $q$. Therefore, $Q_{n+1}$ and $\Delta_{n+1}$ satisfy Property1.

Therefore, we get by construction that $Q_n, \Delta_n$ satisfy Property 1. We define $Q' = \bigcup_{n \geq 0} Q_n$, and $\Delta'' = \bigcup_{n \geq 0} \Delta_n$. Now, using Property 1, $Q' \subseteq Q \cup Q_{\blacklozenge}$ and $\Delta'' \subseteq \Sigma$. This implies that $Q'$ and $\Delta''$ are finite sets. (Note that this means that the series $(Q_n, \Delta_n)$ is eventually constant). Using Property 1, we can easily show that $(Q', \Delta'')$ is a $(P, \Theta)-$ bounded context, and $\Gamma; \Delta'' \nvdash^{Q'} \psi$ ***at*** $q$.

Finally, we define $\Delta'$ to be the set of all sentences $\phi$ ***at*** $s \in \Sigma$ such that $\Gamma; \Delta'' \vdash^{Q'} \phi$ ***at*** $s$. As a consequence of Proposition 25, we get that

$$\Gamma; \Delta' \vdash^{Q'} \mu \textbf{ at } r \text{ if and only if } \Gamma; \Delta'' \vdash^{Q'} \mu \textbf{ at } r \tag{6}$$

Clearly, $\Delta'$ extends $\Delta''$ and hence $\Delta$. Furthermore, $(Q', \Delta')$ is $(P, \Theta)-$bounded by construction. Also we get $\Gamma; \Delta' \nvdash^{Q'} \psi$ ***at*** $q$, thanks to the equivalence (6). We only need to show that $(Q', \Delta')$ is $\Gamma$-prime.

1. (Deductive Closure) The set $\Delta'$ is deductively closed, by construction.
2. (Disjunction Property) Assume that $\Gamma; \Delta' \vdash^{Q'} \psi_1 \vee \psi_2$ ***at*** $r$, for $\psi_1 \vee \psi_2 \in \Theta'$ and $q \in Q'$. Then let $n$ be the least number such that $\Gamma; \Delta_n \vdash^{Q_n} \psi_1 \vee \psi_2$ ***at*** $r$. Clearly, $\psi_1 \vee \psi_2$ ***at*** $q \in treated^{\vee}_n$, and $\Gamma; \Delta_m \vdash^{Q_m} \psi_1 \vee \psi_2$ ***at*** $q$ for every $m \geq n$. Eventually $\psi_1 \vee \psi_2$ ***at*** $q$ has to be treated at some odd stage $h \geq n$. Hence, either $\psi_1$ ***at*** $r \in \Delta_{h+1}$ or $\psi_2$ ***at*** $r \in \Delta_{h+1}$. Therefore, $\psi_1$ ***at*** $q \in \Delta'$ or $\psi_2$ ***at*** $q \in \Delta'$.
3. (Diamond Property) Assume that $\Gamma; \Delta' \vdash^{Q'} \blacklozenge\phi$ ***at*** $r$, for $\blacklozenge\phi \in \Theta'$ and $r \in Q'$. Then let $n$ be the least number such that $\Gamma; \Delta_n \vdash^{Q_n} \blacklozenge\phi$ ***at*** $r$. As in the previous case, we assert that $\blacklozenge\phi$ ***at*** $q$ is treated for some even number $h \geq n$. We get $\phi$ ***at*** $q_{\blacklozenge\phi} \in \Delta'$ by construction.
4. (Consistency) If $\Gamma; \Delta' \vdash^{Q'} \bot$ ***at*** $r$, then $\Gamma; \Delta' \vdash^{Q'} \psi@q$ ***at*** $r$ by the rule $\bot E$. Therefore, $\Gamma; \Delta' \vdash^{Q'} \psi$ ***at*** $q$ by $@E$, which contradicts our construction. Hence, $\Gamma; \Delta' \nvdash^{Q'} \bot$ ***at*** $q$.

We conclude that $(Q', \Delta')$ is a $\Gamma$-prime and $(P, \Theta)-$ bounded context extending $(Q, \Delta)$ such that $\Gamma; \Delta \nvdash^{Q'} \phi$ ***at*** $p$. ∎

We finally construct the bounded canonical model. In the model, the set of Kripke states is the set of prime bounded contexts $(Q, \Delta)$ ordered by inclusion. A place belongs to the state $(Q, \Delta)$ only if it is in $Q$, and an atom $A$ is placed in a place $r$ in the state $(Q, \Delta)$ only if $A$ ***at*** $r \in \Delta$. More formally, we have

**Definition 27 (Bounded Canonical Model)**

2. if $\models_{K}$ is the forcing relation on $K_{can}$, then for every $\phi \in \Theta^{\Box}$, every $(Q, \Delta) \in K$, and every $q \in Q$ it holds: $(Q, \Delta) \models_{K} \phi$ ***at*** $q$ if and only if $\phi$ ***at*** $q \in \Delta$.

**Proof** Clearly, all the properties required for a Kripke model are verified. All we have to prove is the part *2* of the lemma. The proof is standard, and we proceed by induction on the structure of the formula $\phi \in \Theta^{\Box}$. In the induction hypothesis, we assume that part *2* of the lemma is valid on all sub-formulae of $\phi$ that are in $\Theta^{\Box}$. Please note that if $\phi \in \Theta^{\Box}$, then all of the sub-formulae of $\phi$ are in $\Theta^{\Box}$. Hence, we can apply the induction hypothesis on all the sub-formulae of $\phi$. Here, we just illustrate the inductive case in which $\phi$ is $\Box\phi_1$.

*Case* $\Box\phi_1$. Assume that $(Q, \Delta) \models_{K} \Box\phi_1$ ***at*** $q$, where $\Box\phi_1 \in \Theta^{\Box}$. By definition, this means that for every $(Q^{\Box}, \Delta^{\Box}) \geq (Q, \Delta)$ and every $r \in Q^{\Box Q}$

Furthermore, $\Delta$ is contained in $\Sigma$. Therefore, by Lemma 28, $(Q,\Sigma) \models_K \mu$ ***at*** $q$ whenever $\mu$ ***at*** $q \in \Delta$.

Hence, we get that the Kripke state $(Q,\Sigma) \models \Gamma;\Delta$. By our assumption, we get $(Q,\Sigma) \models_K \phi$ ***at*** $p$ also. By Lemma 28, we get $\phi$ ***at*** $p \in \Sigma$. However our choice of $Q,\Sigma$ was such that $\phi$ ***at*** $p \notin \Sigma$. We have just reached a contradiction, and hence we can conclude that $\Gamma;\Delta \vdash^P \phi$ ***at*** $p$. □

Now, by the encoding of Kripke models into birelational models (see Proposition 21), if a judgement is valid in all birelational models then it is valid in all Kripke models. As the class of Kripke models is complete, we get that the class of birelational models is also complete for the logic.

**Corollary 30** If $P$ is finite and the judgement $\Gamma;\Delta \vdash^P \phi$ ***at*** $p$ is bi-valid in every birelational model, then it is provable in the logic.

**Proof** Suppose that the judgement $\Gamma;\Delta \vdash^P \phi$ *at juno-325(tho)15(v)25(able)-255(th)-301(the)-325(6(l(odec.)]T-536he* 6l00f250(loe)-250(l602nite)-2063tndel,pro,.for thmplete contain infinite many worlds. However, by using techniques similar to those used in [35], we will be able to construct a finite model that is equivalent to the counter-model. The key technique in the construction is the identification of triples $(Q, \Delta, q)$ that differ only in renaming of places other than those in $P$. We start the proof by discussing *renaming functions*.

### 5.1 Renaming functions

First, we discuss renaming of places in formulae and judgements. Given any two sets of places $Q_1, Q_2$, a *renaming function* is a function $f : Q_1 \to Q_2$. Intuitively, $f$ renames a place $q$ in $Q_1$ as $f(q)$.

Given a renaming function $f : Q_1 \to Q_2$, we can extend $f$ to a function from the set $Frm(Q_1)$ into the set $Frm(Q_2)$ by replacing all occurrences of places $q$ by $f(q)$. More formally,

- $f(A) \stackrel{\text{def}}{=} A$ for all atoms $A$;
- $f(\phi_1 \circ \phi_2) \stackrel{\text{def}}{=} f(\phi_1) \circ f(\phi_2)$ for $\circ \in \{\wedge, \vee, \to\}$;
- $f(\phi@q) \stackrel{\text{def}}{=} f(\phi)@f(q)$;
- $f(\blacklozenge\phi) \stackrel{\text{def}}{=} \blacklozenge f(\phi)$ and $f(\Box\phi) \stackrel{\text{def}}{=} \Box f(\phi)$.

This can be further extended to contexts $\Gamma; \Delta$ by applying $f$ to all formulae in $\Gamma$ and all sentences in $\Delta$, with $f$ extended to sentences as $f(\phi \ \textbf{\textit{at}}\ q) \stackrel{\text{def}}{=} f(\phi)\ \textbf{\textit{at}}\ f(q)$.

If $f$ is a renaming function, then we can transform a proof of a judgement $\Gamma; \Delta \vdash^{Q_1} \phi\ \textbf{\textit{at}}\ q$ to a proof of the judgement $f(\Gamma; \Delta) \vdash^{Q_2} f(\phi)\ \textbf{\textit{at}}\ f(q)$:

**Lemma 32 (Provability Preservation Under Renaming)** Let $f : Q_1 \to Q_2$ be a renaming function. Then for any set of pure formulae $\Gamma$, any set of sentences $\Delta$, any formula $\phi$ and any place $q$ such that $\mathsf{PL}(\Gamma) \cup \mathsf{PL}(\Delta) \cup \mathsf{PL}(\phi) \cup \{q\} \subseteq Q_1$, we have:

$$\Gamma; \Delta \vdash^{Q_1} \phi\ \textbf{\textit{at}}\ q \text{ implies } f(\Gamma; \Delta) \vdash^{Q_2} f(\phi)\ \textbf{\textit{at}}\ f(q).$$

**Proof** Intuitively, in order to obtain a proof of $f(\Gamma; \Delta) \vdash^{Q_2} f(\phi)\ \textbf{\textit{at}}\ f(q)$, replace all occurrences of places $r$ in the proof of $\Gamma; \Delta \vdash^{Q_1} \phi\ \textbf{\textit{at}}\ q$ by $f(r)$.

More formally, we prove the lemma by induction on $n$, the number of inference rules applied to derive the judgement $\Gamma; \Delta \vdash^{Q_1} \phi\ \textbf{\textit{at}}\ q$. Please note that the induction is on the number of inference rules applied, and we will vary the sets $Q_i, \Delta$, and the formula $\phi$ in the proof. Please recall that the inference rules are given in Fig. 1.

*Base Case ($n = 1$).* Then the rule applied is one amongst $L$, $G$, and $\Box I$. If the applied rule is $L$, then $\phi\ \textbf{\textit{at}}\ q \in \Delta$. Hence $f(\phi)\ \textbf{\textit{at}}\ f(q) \in f(\Delta)$. An application of the rule $L$ gives us $f(\Gamma; \Delta) \vdash^{Q_2} f(\phi)\ \textbf{\textit{at}}\ f(q)$. The cases of $G$ and $\Box I$ follow immediately.

*Induction hypothesis ($n > 1$).* We proceed by cases, and consider the last rule applied to obtain $\Gamma; \Delta \vdash^{Q_1} \phi\ \textbf{\textit{at}}\ q$. The treatment of the rules involving the logical connectives is fairly straightforward, and we show the three most interesting cases: $@I$, $\Box I$, and $\blacklozenge E$.

$@I$: Assume that the last rule applied is $@I$. Then $\phi = \psi@r$, for some pure formula $\psi \in Frm(Q_1)$ and some place $r \in Q_1$. Furthermore, $\Gamma; \Delta \vdash^{Q_1} \psi\ \textbf{\textit{at}}\ p$ is derivable by using less than $n$ instances of the rules.

The induction hypothesis says that $f(\Gamma; \Delta) \vdash^{Q_2} f(\psi)\ \textbf{\textit{at}}\ f(r)$. Using the rule $@I$, we get $\Gamma; \Delta \vdash^{Q_2} f(\psi)@$

The induction hypothesis says that $f'(\Gamma;\Delta) \vdash^{Q_2+q_2'} f'(\psi)$ ***at*** $q_2'$. As $\Gamma, \Delta$ and $\psi$ do not contain $q_1'$, we have $f'(\Gamma;\Delta) = f(\Gamma;\Delta)$ and $f'(\psi) = f(\psi)$. Therefore, by using the inference rule $\Box I$, we get $f(\Gamma;\Delta) \vdash^{Q_2} \Box f(\psi)$ ***at*** $f(q)$. We conclude by observing that $f(\Box\psi) = \Box f(\psi)$.

$\blacklozenge E$: Assume that the last rule applied is $\blacklozenge E$. Then $\phi = \blacklozenge\psi$ for some pure formula $\psi \in Frm(Q_1)$. Moreover, there exist $q_1' \in Q_1$, $q_1'' \in Q_1$, and $\mu \in Frm(P)$ such that:

- $\Gamma;\Delta \vdash^{Q_1} \blacklozenge\mu$ ***at*** $q_1''$ is derivable by using less than $n$ instances of inference rules; and
- $\Gamma;\Delta,\mu$ ***at*** $q_1' \vdash^{Q_1+q_1'} \psi$ ***at*** $q$ is derivable by using less than $n$ instances of inference rules.

By induction hypothesis on the first judgement, we get $f(\Gamma;\Delta) \vdash^{Q_2} \blacklozenge f(\mu)$ ***at*** $f(q_1'')$.

Now, let $Q_1' = Q_1 \cup \{q_1'\}$ and $\Delta' = \Delta \cup \{\mu$ ***at*** $q_1'\}$. We choose $q_2' \notin Q_2$. We define $f' : Q_1' \to Q_2'$ as $f'(r) = f(r)$ for $r \in Q_1$, and $f'(q_1') = q_2'$.

By induction hypothesis on the second judgement, we get $f'(\Gamma;\Delta,\mu$ ***at*** $q_1') \vdash^{Q_2+q_2'} f'(\psi)$ ***at*** $f'(q)$. Now, $f'$ is the same as $f$ on $Q_1$, and therefore $f'(\Gamma;\Delta,\mu$ ***at*** $q_1') = f(\Gamma;\Delta), f(\mu)$ ***at*** $q_2'$ by definition. Hence, we get that $f(\Gamma;\Delta), f(\mu)$ ***at*** $q_2' \vdash^{Q_2+q_2'} f(\psi)$ ***at*** $q$.

We conclude $f(\Gamma;\Delta) \vdash^{Q_2} f(\psi)$ ***at*** $f(q)$, by using the inference rule $\blacklozenge E$. ∎

For example, let us consider $Q_1 = \{p, q\}$ and let $Q_2 = \{r\}$. Let $f : Q_1 \to Q_2$ be the function $f(p) = r$, $f(q) = r$. Let $A$ be an atom, and let $\Gamma$ to be the empty list. We have $\Gamma; A$ ***at*** $p \vdash^{Q_1} A@p$ ***at*** $q$. Then by the Lemma 32, $\Gamma; A$ ***at*** $r \vdash^{Q_2} A@r$ ***at*** $r$.

### *5.2 Pointed Contexts and Morphisms*

Let $P, Q$ be a finite sets of places such that $P \subseteq Q$. Let $\Theta \subseteq Frm(P)$ be a finite set of pure formulae with sub-formula closure $\Theta^*$. Please recall that given a finite set of sentences $\Delta$, we say that $(Q,\Delta)$ is a $(P,\Theta)-$bounded context if for every sentence $\phi$ ***at*** $r$ it is the case that $\phi \in \Theta^*$ and $r \in Q$. Given a $(P,\Theta)-$bounded context $(Q,\Delta)$, we will say that $(Q,\Delta,q)$ is a *pointed* $(P,\Theta)-$*bounded context* if $q \in Q$. Henceforth, we refer to such triples as $(P,\Theta)-$*pcontexts*3The e1f 4 9.963 Tf 17.212 053 9.963 T 9.963 Tf 7.206 3.617 Td[(() 9.963 T 9.963 Tf 7.206 3.617 T

Clearly, $\preceq$ is a preorder. The identity function gives reflexivity, and function composition gives transitivity. This makes the relation $\equiv$ an equivalence relation. If $w$ is a pcontext, then we shall use $[w]$ to denote the class of the pcontexts equivalent to $w$ with respect to the relation $\equiv$. We shall use these equivalence classes as the worlds of the finite counter-model, and the order amongst the worlds will be given by the preorder $\preceq$. We will now show that the relation $\equiv$ partitions the set of pcontexts into finite number of classes. Please note that it is in this proof, we use the fact that the set $P$ is finite:

**Lemma 34 (Finite Partition)** The set of $(P, \Theta)-$pcontexts is partitioned into a finite number of equivalence classes by the equivalence $\equiv$.

**Proof** We will show that every $(P, \Theta)-$pcontext is equivalent to a *canonical pcontext*. The set of canonical pcontexts will be finite. Before we proceed, please note that $P$ and $\Theta$ are finite sets by definition. Hence, the sub-formula closure $\Theta^*$ and the powerset $Pow(\Theta^*)$ must be finite sets.

We will now define the set of canonical pcontexts. For each $\Lambda \subseteq \Theta^*$ we choose a new place $\boldsymbol{r}_{\Lambda} \notin P$ such that $\boldsymbol{r}_{\Lambda_1} \neq \boldsymbol{r}_{\Lambda_2}$ if $\Lambda_1 \neq \Lambda_2$. Let $R \stackrel{\text{def}}{=} \{\boldsymbol{r}_{\Lambda} : \Lambda \subseteq \Theta^*\}$. The cardinality of $R$ is the same as the cardinality of $Pow(\Theta^*)$, and hence $R$ is finite. A canonical pcontext will have places amongst $P \cup R$. Furthermore, the canonical pcontext will contain the sentence $\phi$ ***at*** $\boldsymbol{r}_{\Lambda}$ if and only if $\boldsymbol{r}_{\Lambda}$ is a place in the pcontext and $\phi \in \Lambda$. More formally, we say that the triple $(Q, \Sigma, q)$ is a *canonical* $(P, \Theta)$-*pcontext* if

- $Q$ is a set of places such that $P \subseteq Q \subseteq P \cup R$.
- $\Delta$ is the union of two sets $\Delta_P$ and $\Delta_R$, where
  1. $\Delta_P$ is a set of sentences such that $\phi$ ***at*** $s \in \Delta_P$ means that $\phi \in \Theta^*$ and $s \in P$; and
  2. $\Delta_R$ is the set of *all* sentences $\phi$ ***at*** $\boldsymbol{r}_{\Lambda}$, where $\phi \in \Lambda$ and $\boldsymbol{r}_{\Lambda} \in Q \cap R$. In other words, $\Delta_R \stackrel{\text{def}}{=} \{\phi \text{ at } \boldsymbol{r}_{\Lambda} : \phi \in \Lambda,\ \boldsymbol{r}_{\Lambda} \in Q \cap R\}$.
- $q \in Q$.

Clearly, a triple that satisfies the above points is a $(P, \Theta)-$pcontext. Furthermore, as the sets $P, R, \Theta^*$ are finite, the set of canonical pcontexts must be finite also.

We will now show that for every pcontext $w = (Q, \Delta, q)$ there is a canonical pcontext equivalent to it. This would immediately give us that the number of equivalence classes induced by $\equiv$ is finite.

Let $w = (Q, \Delta, q)$ be a $(P, \Theta)-$pcontext, and fix it. For $s \in Q$, let $H(s) \subseteq \Theta^*$ be the set of formulae $\phi$ such that $\phi$ ***at*** $s \in \Delta$.

We now define $w' = (Q', \Delta', q')$, the canonical pcontext equivalent to $w$ as follows. $P$ will be contained in $Q'$. For each $s \in Q \setminus P$, we add the place $\boldsymbol{r}_{H(s)}$ to $Q'$. For $p \in P$, a sentence $\phi$ ***at*** $p$ will be in $\Delta'$ only if it is in $\Delta$. A sentence $\phi$ ***at*** $\boldsymbol{r}_{H(s)}$ will be [illegible] r4TJ/F79 6.974 Tf 19.2

Clearly, $(Q', \Delta', q')$ is a canonical $(P, \Theta)-$pcontext. Moreover, the renaming functions

$$f : Q \longrightarrow Q' \qquad f(s) \stackrel{\text{def}}{=} \begin{cases} s & \text{if } s \in P; \\ \mathbf{r}_{H(s)} & \text{otherwise.} \end{cases}$$

$$g : Q' \longrightarrow Q \qquad g(t) \stackrel{\text{def}}{=} \begin{cases} t & \text{if } t \in P; \\ q & \text{if } t = q'; \\ l & \text{otherwise, where } l \in Q \setminus P \text{ is chosen s.t.} \\ & t = \mathbf{r}_{H(l)}. \end{cases}$$

are morphisms from $w$ to $w'$ and from $w'$ to $w$, respectively. We conclude that $w \approx w'$. □

### 5.3 The Finite Counter-Model

Given a finite set of places $P$, two finite sets of pure formulae $\Gamma, \Theta \subseteq Frm(P)$, let $\mathsf{K}_{can}$ be the $\Gamma-$prime and $(P, \Theta)-$bounded canonical Kripke model as defined in §4 (see Definition 27). Now, let $\mathsf{W}_{can} = (W, \leq, R, I, Eval)$ be the $\mathsf{K}_{can}-$ birelational model obtained by using the encoding of $\mathsf{K}_{can}$ into a birelational model (see §3.2). We call $\mathsf{W}_{can}$ the $\Gamma-$prime and $(P, \Theta)-$bounded canonical birelational model. Please recall from the proof of completeness (see §4) that if a judgement $\Gamma; \Sigma \vdash^{P} \phi$ ***at*** $p$ is not provable, then $\mathsf{W}_{can}$ provides the birelational counter-model for the judgement for an appropriate choice of $\Theta$.

The worlds of $\mathsf{W}_{can}$ are pcontexts $(Q, \Delta, q)$ where $(Q, \Delta)$ are $\Gamma-$prime and $(P, \Theta)-$bounded. Two worlds $w_1 = (Q_1, \Delta_1, q_1)$ and $w_2 = (Q_2, \Delta_2, q_2)$ are reachable from each other if $Q_1 = Q_2$ and $\Delta_1 = \Delta_2$. Furthermore, $(Q_1, \Delta_1, q_1) \leq (Q_2, \Delta_2, q_2)$ if $Q_1 \subseteq Q_2$, $\Delta_1 \subseteq \Delta_2$ and $q_1 = q_2$. A world $w = (Q, \Delta, q) \in I(A)$ for some atom $A$ if $A$ ***at*** $q \in \Delta$. The evaluation is a total function, and $E((Q, \Delta, q)) = q$. Furthermore, as a consequence of definition of canonical models, a world $w = (Q, \Delta, q)$ forces a formula $\phi \in \Theta'$ if and only if $\phi$ ***at*** $q \in \Delta$.

Even though the worlds in canonical birelational are composed of bounded pcontexts, the set of the worlds may itself be infinite. Following [35], we shall construct a model, called the *quotient model*, equivalent to the canonical model. For this model, we will use morphisms between pcontexts. Please recall that given pcontexts $w_1$ and $w_2$, $w_1 \preceq w_2$ if there is a morphism from $w_1$ into $w_2$, and $w_1 \approx w_2$ if $w_1 \preceq w_2$ and $w_2 \preceq w_1$. The relation $\preceq$ is a preorder and $\approx$ is an equivalence. The set of equivalence classes generated by $\approx$ is finite by Lemma 34. We write $[w]$ for the equivalence class of $w$.

In the quotient canonical model, the set of worlds will be $W_{/\approx}$, the set of equivalence classes generated by $\approx$ on $W$. We have that $W_{/\approx}$ is finite. Our construction will ensure that $w$ in the canonical birelational model forces a formula $\phi \in \Theta'$ only if $[w]$ forces $\phi$.

In the quotient model, $[w_1]$ will be less than $[w_2]$ only if $w_1 \preceq w_2$. As $\preceq$ is a preorder, it follows easily that this ordering is well-defined. If $R$ is the reachability relation on the canonical model, then $[w_1]$ is reachable from $[w_2]$ in the classes gener

1. The set $W_{/\sim}$ is the set of the equivalence classes generated by the relation $\sim$ on $W$.

2. The binary relation $\leq^{\sim}$ is defined as: $[w_1] \leq^{\sim} [w_2]$ if and only if $w_1 \sim w_2$.

3. The binary relation $R^{\sim}$ is defined as: $[w_1]\, R^{\sim} [w_2]$ if and only if there exists $w_1' \in [w_1]$ and $w_2' \in [w_2]$ such that $w_1'\, R\, w_2'$.

4. The function $I^{\sim} : Atoms \to Pow(W_{/\sim})$ is defined as:

$$I^{\sim}(A) \stackrel{\text{def}}{=} \{[w] : w \in I(A)\}$$

5. The partial function $Eval^{\sim} : W_{/\sim} \to P$ is defined as:

$$Eval^{\sim}([w]) \stackrel{\text{def}}{=} \begin{cases} p & \text{if } w = (Q, \Delta, p) \text{ and } p \in P; \\ \text{not defined} & \text{otherwise.} \end{cases}$$

As we discussed before, $\leq^{\sim}$, $R^{\sim}$, $I^{\sim}$ and $Eval^{\sim}$ in the quotient model are well-defined. We show that the relation $R^{\sim}$ is an equivalence:

**Lemma 36 (Reachability is an Equivalence)** Given a finite set of places $P$, two finite sets of pure formulae $\Gamma, \Theta \subseteq Frm(P)$, let $\mathsf{W}_{can} = (W, \leq, R, I, Eval)$ be the $\Gamma-$prime and $(P,\Theta)-$bounded canonical birelational model. Let $\mathsf{W}_{/\sim} = (W_{/\sim}, \leq^{\sim}, R^{\sim}, I^{\sim}, Eval^{\sim})$ be the quotient model of $\mathsf{W}_{can}$. Then $R^{\sim}$ is an equivalence.

**Proof** The reflexivity and symmetry of $R^{\sim}$ follow from the reflexivity and symmetry of $R$ in the model $\mathsf{W}_{can}$. We need to show that $R^{\sim}$ is transitive.

Pick $[w_1], [w_2], [w_3] \in W_{/\sim}$ such that $[w_1]\, R^{\sim} [w_2]\, R^{\sim} [w_3]$, and fix them. By definition, the assumption $[w_1]\, R^{\sim} [w_2]\, R^{\sim} [w_3]$ is equivalent to saying that there are $w_1', w_2', w_2'', w_3' \in W$ such that $w_1 \sim w_1'\, R\, w_2' \sim w_2$ and $w_2 \sim w_2''\, R\, w_3' \sim w_3$. As $\sim$ is an equivalence, we get

$$w_1'\, R\, w_2' \sim w_2''\, R\, w_3'. \tag{7}$$

In order to prove transitivity, we will first show that there are two worlds $v_1$ and $v_3$ in $W$ such that $w_1' \sim v_1\, R\, v_3 \sim w_3'$. This will give us by definition $[w_1']\, R^{\sim} [w_3']$, and hence $[w_1]\, R^{\sim} [w_3]$.

Now, the assumptions in (7) and the definition of $R$ say that

1. $w_1' = (Q_1, \Delta_1, q_1)$ and $w_2' = (Q_1, \Delta_1, q_2)$, where $(Q_1, \Delta_1)$ is a $\Gamma$-prime and $(P,\Theta)-$bounded context, and $q_1, q_2 \in Q_1$.

2. $w_2'' = (Q_2, \Delta_2, q_2')$ and $w_3'' = (Q_2, \Delta_2, q_3)$, where $(Q_2, \Delta_2)$ is a $\Gamma$-prime and $(P,\Theta)-$bounded context, and $q_2', q_3 \in Q_2$.

3. $(Q_1, \Delta_1, q_2) \sim (Q_2, \Delta_2, q_2')$, *i.e.*, there exist two morphisms $f : Q_1 \to Q_2$ and $g : Q_2 \to Q_1$ such that $f(q_2) = q_2'$ and $g(q_2') = q_2$.

Without loss of generality, we can assume that $Q_1 = P \uplus R_1$ and $Q_2 = P \uplus R_2$ with $R_1 \cap R_2 = \emptyset$ (otherwise, we can rename the places in $\Delta_2$ and $R_2$).

$(Q_1 \cup Q_2, \Delta_1 \cup \Delta_2)$ is $(P,\Theta)-$bounded as $(Q_1, \Delta_1)$ and $(Q_2, \Delta_2)$ are bounded contexts. We let $v_1 \stackrel{\text{def}}{=} (Q_1 \cup Q_2, \Delta_1 \cup \Delta_2, q_1)$ and $v_3 \stackrel{\text{def}}{=} (Q_1 \cup Q_2, \Delta_1 \cup \Delta_2, q_3)$.

Now, consider the triple $v_1 = (Q_1 \cup Q_2, \Delta_1 \cup \Delta_2, q_1)$. We have $(Q_1 \cup Q_2, \Delta_1 \cup \Delta_2, q_1) \sim (Q_1, \Delta_1, q_1)$, by considering the two renaming functions

$$G_1 : Q_1 \cup Q_2 \longrightarrow Q_1 \qquad G_2 : Q_1 \longrightarrow Q_1 \cup Q_2$$

$$G_1(q) \stackrel{\text{def}}{=} \begin{cases} q & \text{if } q \in Q_1; \\ g(q) & \text{if } q \in Q_2 \end{cases} \qquad G_2(q) \stackrel{\text{def}}{=} q$$

Please note that as $g$ is a morphism, $g(q) = q$ if $q \in Q_1 \cap Q_2 = P$. Therefore, $G_1$ is well-defined and $G_1(q_1) = q_1$. Now, suppose that $\phi$ ***at*** $q \in \Delta_1 \cup$ $q \in$

Similarly, $(Q_1 \cup Q_2, \Delta_1 \cup \Delta_2, q_3) \leq (Q_2, \Delta_2, q_3)$ by considering the morphisms

$$F_1 : Q_1 \cup Q_2 \longrightarrow Q_2 \qquad\qquad F_2 : Q_2 \longrightarrow Q_1 \cup Q_2$$

$$F_1(q) \stackrel{\text{def}}{=} \begin{cases} f(q) & \text{if } q \in Q_1; \\ q & \text{if } q \in Q_2 \end{cases} \qquad\qquad F_2(q) \stackrel{\text{def}}{=} q$$

We get that $v_3 \leq w_3'$.

If $v_1$ and $v_3$ are worlds in $W_{can}$, then $v_1 \, R \, v_3$ by definition. In that case $v_1$ and $v_3$ are the worlds we are looking for. In order to show that $v_1$ and $v_3$ are indeed worlds in $W_{can}$ we need to show that the $(P, \Theta)$−bounded context $(Q_1 \cup Q_2, \Delta_1 \cup \Delta_2)$ is $\Gamma$-prime.

In order to show that $(Q_1 \cup Q_2, \Delta_1 \cup \Delta_2)$ is $\Gamma$−prime we need to show the four properties required by Definition 24. We will prove here only the $\Theta$-deductive closure property. The treatment of other properties is similar.

Assume that $\Gamma; \Delta_1 \cup \Delta_2 \vdash^{Q_1 \cup Q_2} \phi \, \textbf{\textit{at}} \, q$ for some $\phi \in \Theta$. We consider two cases. If $q \in Q_1$, then consider the renaming function $G_1$ defined above. Now $G_1$ fixes $Q_1$ and morphism from $w_1$ to $w_2$ that fixes $q$. Therefore, $w_2 = (Q_2, \Delta_2, q)$ for some $Q_2$ and $\Delta_2$. By definition, we conclude that $[w_2]\downarrow q$.

*Uniqueness* Consider $[w_1], [w_2] \in W_{/\equiv}$ such that $[w_1]\, R^{\equiv} [w_2]$. This means that there exist $w_1', w_2' \in W$ such that $w_1 \equiv w_1'\, R\, w_2' \equiv w_2$. Assume that $[w_1]\downarrow q$ and $[w_2]\downarrow q$. Then $w_1'\downarrow q$ and $w_2'\downarrow q$ in $\mathsf{W}_{can}$. The uniqueness property in $\mathsf{W}_{can}$ says that $w_1' = w_2'$. Hence $w_1 \equiv w_1' \equiv w_2$. We conclude $[w_1] = [w_2]$ as required. $\square$

We will show that a world $w$ forces a formula in $\Theta^{*}$ in the canonical birelational model if and only if $[w]$ forces the formula in the quotient model. For this, we will need the following proposition which states that given worlds $w_1 \preceq w_2$ in the canonical model, if $w_1$ forces a formula in $\Theta^{*}$ then so does $w_2$:

**Proposition 38 (Forcing Preservation Under Morphisms)** Given a finite set of places $P$, two finite sets of pure formulae $\Gamma, \Theta \subseteq Frm(P)$, let $\mathsf{W}_{can} = (W, \leq, R, I, Eval)$ be the $\Gamma-$prime and $(P, \Theta)-$ bounded canonical birelational model. Let $\models_{\mathsf{W}}$ be the extension of interpretation $I$ to formulae. Then for every $w_1, w_2 \in W$, and $\phi \in \Theta^{*}$:

1. If $w_1 \preceq w_2$, then $w_1 \models_{\mathsf{W}} \phi$ implies $w_2 \models_{\mathsf{W}} \phi$.
2. If $w_1 \equiv w_2$, then $w_1 \models_{\mathsf{W}} \phi$ if and only if $w_2 \models_{\mathsf{W}} \phi$.

**Proof** We prove the first point as the second one is straightforward consequence of the first one. Consider $w_1, w_2 \in W$, such that $w_1 \preceq w_2$. This means that $w_1 = (Q_1, \Delta_1, q_1)$ and $w_2 = (Q_2, \Delta_2, q_2)$ where $(Q_i, \Delta_i)$ are $\Gamma$-prime and $(P, \Theta)$-bounded contexts for $i = 1, 2$. Moreover, there is a morphism $f : Q_1 \to Q_2$ such that $f(q_1) = q_2$.

Assume that $w_1 \models_{\mathsf{W}} \phi$ for some $\phi \in \Theta^{*}$. This means from the definition of canonical birelational model that $\phi\ \boldsymbol{at}\ q_1 \in \Delta_1$. Since $f$ is a morphism from $w_1$ to $w_2$, we get that $\phi\ \boldsymbol{at}\ q_2 \in \Delta_2$. Once again, we get from the definition of canonical birelational model that $w_2 \models_{\mathsf{W}} \phi$. $\square$

We are now ready to prove that if the world $w$ in the canonical birelational model forces $\phi \in \Theta^{*}$, then the world $[w]$ in the quotient model also forces $\phi$, and vice-versa.

**Lemma 39 (Quotient Forcing Preservation)** Given a finite set of places $P$, two finite sets of pure formulae $\Gamma, \Theta \subseteq Frm(P)$, let $\mathsf{W}_{can} = (W, \leq, R, I, Eval)$ be the $\Gamma-$prime and $(P, \Theta)-$ bounded canonical birelational model. Let $\mathsf{W}_{/\equiv} = (W_{/\equiv}, \leq^{\equiv}, R^{\equiv}, I^{\equiv}, Eval^{\equiv}$

$\phi_2$. The induction hypothesis says that $[w'] \models_{/□} \phi_2$. As $[w']$ is an arbitrary world larger that $[w]$, we can conclude that $[w] \models_{/□} \phi_1 \to \phi_2$.

For the other direction, let $[w] \models_{/□} \phi$. This means that for every $[w'] \geq' [w]$: if $[w'] \models_{/□} \phi_1$, then $[w'] \models_{/□} \phi_2$.

Consider now $w' \geq' w$. We have $[w']$ □ $[w]$ also. If we assume $w' \models_W \phi_1$, then the induction hypothesis says that $[w'] \models_{/□} \phi_1$. Then $[w'] \models_{/□} \phi_2$, and so $w' \models_W \phi_2$ by induction hypothesis. We conclude that $w \models_W \phi_1 \to \phi_2$.

Case $\phi = \Box\phi_1$. Let $w \models_W \phi$. We need to show that $[w] \models_{/□} \Box\phi_1$. Consider $[w_1] \geq' [w]$ and $[w_2]\, R'[w_1]$. It suffices to show that $[w_2] \models_{/□} \phi_1$. The hypothesis $[w_2]\, R'[w_1] \geq' [w]$ means that $w_1$ □ $w$ and $w_2$ □ $w_3\, R\, w_4$ □ $w_1$ for some worlds $w_3, w_4$ □ $W$. We get that $w_4$ □ $w$ as □

**Corollary 41 (Decidability)** The provability of the judgement $\Gamma; \Delta \vdash^{P} \phi$ ***at*** $p$ is decidable in the logic.

**Proof** Let $P'$ be $\mathsf{PL}(\Gamma) \cup \mathsf{PL}(\Delta) \cup \mathsf{PL}(\phi) \cup \{p\}$. By Proposition 31, $\Gamma; \Delta \vdash^{P} \phi$ ***at*** $p$ if and only if $\Gamma; \Delta \vdash^{P'} \phi$ ***at*** $p$. As the function $\mathsf{PL}$ can be effectively computed, we just need to consider the judgement $\Gamma; \Delta \vdash^{P'} \phi$ ***at*** $p$ for the decidability result.

We can enumerate all proofs in the logic in which the set of places considered is finite. Hence, we obtain an effective enumeration of all provable judgements. We can also effectively enumerate all finite birelational models, and effectively check whether the judgement $\Gamma; \Delta \vdash^{P'} \phi$ ***at*** $p$ is refutable in a given finite birelational model. As a consequence of the finite model property proved above, $\Gamma; \Delta \vdash^{P'} \phi$ ***at*** $p$ is refutable only if it is refutable in some finite birelational model. By performing these enumerations and checks simultaneously, we obtain an effective test for provability of $\Gamma; \Delta \vdash^{P'} \phi$ ***at*** $p$. □

The procedure detailed in the corollary above would not have worked if we had used Kripke models instead of birelational models. This is because the finite model property fails for Kripke models. For example, consider the judgement $; \Box\neg\neg A$ ***at*** $p \vdash^{\{p\}} \neg\neg\Box A$ ***at*** $p$. We claim that this judgement is valid for every *finite* Kripke model.

Indeed, let $k$ be a Kripke state in some finite Kripke model $\mathsf{K}$ such that $(k, p) \models \Box\neg\neg A$. Pick $l \geq k$ in $\mathsf{K}$ such that $l$ is maximal with respect to the ordering of Kripke states. As $(k, p) \models \Box\neg\neg A$, we get by definition that $(l, r) \models \neg\neg A$ for every place $r$ in the state $l$. From the semantics of implication and the fact that $l$ is a maximal state, it must be the case that $(l, r) \models A$ for every place $r$ in the state $l$. Again, as $l$ is maximal, we get $(l, p) \models \Box A$, and therefore $(l, p) \models \neg\neg\Box A$. As the model is finite, there is always a maximal $l$ above any $k' \geq k$, and then $(l, p) \models \Box A$. We conclude $(k, p) \models \neg\neg\Box A$.

On the other hand, we showed that the judgement is not valid in the finite model $\mathsf{W}_{exam}$ in Ex. 11. The model $\mathsf{W}_{exam}$ has two worlds $w_1$ and $w_2$ such that $w_1 \leq w_2$, $w_1 \, R \, w_2$, $I(A) = \{w_2\}$, $w_1\!\uparrow$ and $w_2\!\downarrow p$. As we discussed there, $w_2 \models \Box\neg\neg A$ and $w_2 \not\models \neg\neg\Box A$. As we mentioned before, this example is adapted from [24, 35].

## 6 Related Work

The logic we studied is an extension of the logic introduced in [16, 17]. In [16, 17], it was used as the foundation of a type system for a distributed $\lambda$-calculus by exploiting the *proofs-as-terms and propositions-as-types* paradigm. The proof terms corresponding to modalities have computational interpretation in terms of remote procedure calls ($@p$), commands to broadcast computations ($\Box$), and commands to use portable code ($\blacklozenge$). The authors also introduce a sequent calculus for the logic without disjunctive connectives, and prove that it enjoys cut elimination. Although the authors demonstrate the usefulness of logic in reasoning about the distribution of resources, they do not have a corresponding model.

The *proofs-as-terms and propositions-as-types* paradigm has also been used in [37, 38, 21]. In [37], the logic studied is an intuitionistic modal logic derived from *IS5*, and the modalities have a spatial flavour. Specifically, Kripke states are taken to be nodes on a network. The connective $\Box$ refelects the mobility of portable code, and $\blacklozenge$ reflects the address of a fixed resources. The work in [38] extends [37, 16, 17] to a lambda calculus for classical hybrid *S5* with network-wide continuations, which arise naturally from the underlying classical logic. These continuations create a new relationship between the two modalities and give a computational interpretation of theorems of classical hybrid *S5*. In [21], the relationship modal logics and type systems for Grid computing is investigated. The objects with type $\Box$ are interpreted as jobs that may be injected into the Gird and run anywhere. The main difference from [38, 37, 16, 17] is that the underlying logic is based on *S4* rather than *S5*. Whereas [38, 37, 16, 17] assume all nodes are connected to all other modalities built from pure names. The original idea of internalising the model into formulae was proposed in [27, 28], and has been further investigated in [1, 2, 4, 5, 6]. This work has been mostly carried out in the classical setting. More recently, classical hybrid logic is combined with linear temporal logic in [25], and the logic accounts for both temporal and spatial aspects. Intuitionistic versions of hybrid logics were investigated in [7, 16, 17].

There are several intuitionistic modal logics in the literature, and [35] is a good source on them. The modalities in [35] have a temporal flavour, and the spatial interpretation was not recognised then. In [35], for example, the accessibility relation expresses the next step of a computation. The work in [7] extends the modal systems in [35], and creates hybrid versions of the modal systems by introducing *nominals*, a new kind of propositional symbols projecting semantics into the logic. A natural deduction system for these hybrid systems along with a normalisation result is also given in [7]. A Kripke semantics along with a proof of soundness and completeness is also introduced.

The extension we gave to the logic in [16, 17] is a hybrid version of the intuitionistic modal system *IS5* [23, 29, 35]. The modality $@p$ internalises the model in the logic. In the modal system *IS5*, first introduced in [29], the accessibility relation among places is total. The main difference in the logic presented in [7] and the logic in [16, 17] is that names in [16, 17] only occur in the modality $@p$.

From the point of view of semantics, Kripke semantics were first introduced in [19] for intuitionistic first-order logic. Kripke semantics for intuitionistic modal systems were developed in [11, 23, 26, 34, 35]. Birelational models for intuitionistic modal logic were introduced independently in [11, 34, 26]. They are in general useful to prove the finite model property as demonstrated in [24, 35]. The finite model property fails for Kripke semantics [35, 24], and an example for this was adapted in this paper.

Some other examples of work on logics for resources are separation logics [33] and **BI**, the logic of bunched implications [22, 31, 32]. Separation logic is an extension of Hoare logic that permits reasoning about low-level imperative programs with shared mutable data structure. Formulae are extended by introducing a 'separating conjunction' whose subformulae are meant to hold for disjoint parts of the system, thus enabling a concise and flexible description of structures with controlled sharing. **BI** is the theoretical base to separation logics. While separation logic is based on particular storage models, **BI** describe resources more generally and its model theory is inspired by a primitive of resource composition.

The logic of bunched implications is a substructural system which freely combines propositional intuitionistic logic and the multiplicative fragment of propositional linear logic. Assertions are not in a sequence, but rather in *bunches*: contexts with two combining operations, one reflected in the logic the intuitionistic conjunction and the other by the multiplicative one. In [22, 31, 32], the authors give a Kripke model based on monoids. The formulae of the logic are the resources, and are interpreted as elements of the monoid. The monoidal operation is reflected in the logic by the multiplicative connective. The focus of this work is the sharing of resources, and not their distribution.

**BI-Loc**, presented in [3], extends the logic of bunched implication by introducing a modality for locations. Its models are *resource trees*: node-labelled trees in which nodes contain resources belonging to a monoid. Every label gives rise to a corresponding logical modality which precisely indicates the location where a formula holds. Although **BI-Loc** offers a separation operator to express properties holding in different parts of the system, its o2i002ed inn fpitsinn unspec7S1 the asynchronous $\pi$-calculus [20]. The logic is developed in classical settings and lacks a notion of resources. The main aim of spatial logic is to describe the behaviour and the spatial structure of concurrent systems. The logic is modal in space and in time, and a formula describes a property of a particular part of a concurrent system at a particular time.

Locations can be added to Spatial Logic along the lines of [9] which gives a modal logic based on Ambient Calculus [10]. Ambients are intended as locations, and there is a modality $m\ [\_]$ for every ambient name $m$ which specifies the location where a property holds. These spatial modalities have an intensional flavour and 'hybridise' spatial logics as the modality $@p$ 'hybridises' *IS5* in the current paper. However, the locations in Ambient logic unlike this paper have an intensional hierarchy which is reflected in the logic by having nested formulae like $m\ [n\ [\Box]]$.

## 7 Conclusions and Future Work

We studied the hybrid modal logic presented in [16, 17], and extended the logic with disjunctive connectives. Formulae in the logic contain names, also called places. The logic is useful to reason about placement of resources in a distributed system. We gave two sound and complete semantics for the logic.

In one semantics, we interpreted the judgements of the logic over Kripke-style models [19]. Typically, Kripke models [19] consist of partially ordered Kripke states. In our case, each Kripke state has a set of places, and different places satisfy different formulae. Larger Kripke states have larger sets of places, and the satisfaction of atoms corresponds to the placement of resources. The modalities of the logic allow formulae to be satisfied in a named place ($@p$), some place ($\blacklozenge$) and every place ($\Box$). The Kripke semantics can be seen as an instance of hybrid *IS5* [23, 29, 7, 35].

In the second semantics, we interpreted the judgements over birelational models [11, 34, 26, 35]. Typically, birelational models have a set of partially ordered worlds. In addition to the partial order, there is also a reachability relation amongst worlds. In order to interpret the modality $@p$ in the system, we also introduced a partial evaluation function on the set of worlds. The hybrid nature of the logic presented difficulties in the proof of soundness. The difficulties are addressed using a mathematical construction that creates a new model from a given one. The set of worlds in the constructed model is the union of two sets. One of these sets is the reachability relation, and the worlds in the second set witness the existential and universal properties.

As in the case of intuitionistic modal systems [11, 34, 23, 26, 35], we demonstrated that the birelational models introduced here enjoy the finite model property: a judgement is not provable in the logic if and only if it is refutable in some finite model. The finite model property allowed us to conclude decidability. The partiality of the evaluation function was essential in the proof of the finite model property.

As future work, we are considering other extensions of the logic. A major limitation of the logic presented in [16, 17] is that if a formula $\phi$ is validated at some named place, say $p$, then the formula $\phi@p$ can be inferred at every other place. Similarly, if $\blacklozenge\phi$ or $\Box\phi$ can be inferred at one place, then they can be inferred at any other place. In a large distributed system, we may want to restrict the rights of accessing information in a place. This can be done by adding an accessibility relation as is done in the case of other intuitionistic modal systems [35, 7]. We are currently investigating if the proof of the finite model property can be adapted to the hybrid versions of other intuitionistic modal systems. We are also investigating the computational interpretation of these extensions. This would result in extensions of $\lambda$-calculus presented in [16, 17]. We also plan to investigate adding temporal modalities to the logic. This will help us to reason about both space and time.

From a purely logical point of view, the meta-logic used in the paper to reason about soundness and completeness is classical. In order to obtain a full intutionistic account for the logic, another line of investigation would be to consider categorical and/or topological semantics for the logic. This would allow us to obtain soundness and completeness results when the meta-logic is intuitionistic.

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